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Abstract. From time to time modifications of Einstein's general theory of relativity have been considered the central feature of which, on a formal level, is the appearance in place of the Einstein Lagrangian R of a more general Lagrangian L. On physical grounds one will require L to differ from R only by terms whose presence has no substantial observational consequences under 'ordinary' conditions. No non-trivial exact vacuum solutions of the equations generated by any such Lagrangian appear to be known, the Schwarzschild metric being regarded as a 'trivial solution' in this context. Here I consider pp waves, first as exact solutions of the vacuum equations generated by a family of inhomogeneous quadratic Lagrangians L^*. Thereafter a much wider class of Lagrangians is contemplated, but the greater generality of this is only apparent since all such Lagrangians in effect reduce to L^*.

1. Introduction

Since the early days of general relativity theory alternative theories have been proposed for various reasons (see e.g. Stelle 1978) in which the energy–momentum tensor is equated to the functional derivative of an invariant of the Riemann tensor other than the scalar curvature R, the latter choice being central to Einstein's theory. I shall call any such Lagrangian L^* 'nonlinear' since it is not a linear function of the components of the Riemann tensor. In the context of investigations devoted to finding exact solutions of the vacuum field equations generated by some given L^*, attention has hitherto been confined almost entirely to the class of homogeneous quadratic invariants. The members of this class are, in the first instance, linear sums with constant coefficients of the four 'elementary quadratic invariants' $K_1 := R^2$, $K_2 := R_{ij}R^{ij}$, $K_3 := R_{ijkl}R^{ijkl}$, $K_4 := \overset{+}{R}_{ijkl}R^{ijkl}$. (It is taken for granted that the Riemann space is four dimensional.) Since the functional derivatives of $K_1 - 4K_2 + K_3$ and of K_4 vanish identically, one is left in effect with the two-parameter family

$$L^*(\beta, \gamma) = \beta R^2 + \gamma R_{ij}R^{ij} \qquad (\beta, \gamma = \text{constant}). \tag{1.1}$$

As far as I am aware almost all the known exact solutions of the vacuum equations $\delta L^*(\beta, \gamma)/\delta g_{ij} =: P^{ij}(\beta, \gamma) = 0$ are not of any genuine interest: when $\gamma = 0$ the equations are satisfied by any V_4 whose scalar curvature is constant; whatever the values of α and β may be, they are satisfied when the V_4 is an arbitrary Einstein space; when $3\beta + \gamma = 0$ they are satisfied by any V_4 conformal to an arbitrary Einstein space (Buchdahl 1953). Such solutions are evidently either too general, or else do not go beyond those familiar from Einstein's theory. The only known vacuum solution to which this remark does not apply belongs to $L^*(1, 0)$ under the assumption that R is not constant (Buchdahl 1978). Information is also available about the question of the existence of static, regular, asymptotically flat solutions of $P^{ij}(\beta, \gamma) = 0$ (Buchdahl 1973). However, all such results are largely irrelevant: homogeneous quadratic Lagrangians must be rejected because they do not lead to acceptable theories of gravitation (Pechlaner and Sexl 1966, Folomeshkin 1971, Havas 1977, Stelle 1978).

Granted, then, that Lagrangians L^* more general than (1.1) need to be considered, I know of only two relevant results which are about exact solutions of the vacuum equations. The first is this: when L^* is an invariant of the Ricci tensor and $L^*(-g)^{1/2}$ is scale invariant, i.e. invariant under the substitution $g_{ij} \rightarrow \sigma g_{ij}$ (σ = constant), the vacuum equations are satisfied by an arbitrary Einstein space (Buchdahl 1948a). The second concerns the exceptional Lagrangian

$$L^* = R - (2K_1 - 4K_2)^{1/2}. \tag{1.2}$$

This is remarkable in that whereas the vacuum field equations which it generates are satisfied by an arbitrary Einstein space, they are also satisfied by a V_4 which is reciprocal to an arbitrary static Einstein space (Buchdahl 1970a). (The term 'reciprocal' is here used in the sense in which it appears in the context of what was historically the first of the 'generation methods' (Buchdahl 1954).) In the first case, then, the known vacuum solutions once again do not go beyond those of Einstein's theory. In the case of (1.2) they do, to the extent that a space reciprocal to a static, non-special Einstein space is not an Einstein space. Nevertheless, I believe that (1.2), like all the other Lagrangians mentioned so far, should be rejected. Einstein's Lagrangian R leads to a theory no predictions of which are in clear disagreement with any observational results available at present. Under the 'ordinary' physical circumstances to which these relate any alternative theory, to the extent that it is based on similar principles (Buchdahl 1981), must therefore 'resemble' Einstein's very closely: to all intents and purposes its Lagrangian L^* must, loosely speaking, differ only insignificantly from R. In other words, writing $L^* =: R + \Lambda$, the effects of those terms of the field equations which are generated by Λ must, under 'ordinary' circumstances, be entirely masked by the effects of terms generated by R. As far as exact solutions are concerned, Lagrangians of this more general kind have only occurred in the context of cosmological theory (Buchdahl 1970b). In general, however, one will wish to confine oneself in the first instance to vacuum solutions, not least to avoid the difficulties engendered by having to take into account the limitations implicit in the physical nature of possible sources; but no such exact vacuum solutions seem to be available.

Reflect now that the relatively simple state of affairs which obtains in the context of the cosmological solution referred to a moment ago is related to the fact that the assumed generic form of the metric—the Robertson–Walker metric—contains only a single function of one variable to be determined by the field equations. This suggests that when seeking vacuum solutions one should choose a generic form of the metric which satisfies two requirements: (i) it shall again involve only a single function to be determined by the field equations and (ii) the algebraic invariants of the Riemann tensor, evaluated for this generic metric, shall have 'as simple a form as possible'. The meaning of the second demand is rather nebulous, but one might not unreasonably take an invariant to have the simplest possible form when it vanishes identically. The specific choice of the generic pp-wave metric (Kramer *et al* 1980)

$$-ds^2 = dx^2 + dy^2 + 2H(x, y, u)\, du + 2\, du\, dv \tag{1.3}$$

so suggests itself, for it contains only the one function H to be determined whilst *all* algebraic invariants of the Riemann tensor vanish, whatever the form of H; as do, in fact, all invariants which can be formed by transvection from the metric tensor, the Riemann tensor and its covariant derivatives (Jordan *et al* 1960).

Next, a Lagrangian needs to be chosen. With tradition in mind, a first choice will be the family of inhomogeneous quadratic Lagrangians

$$L^*(\lambda, \eta, \beta, \gamma) = \lambda + \eta R + \beta R^2 + \gamma R_{ij} R^{ij}, \tag{1.4}$$

where λ, η, β, γ are constants, with $\eta = 0$ or 1. (The possibility of η having the value 0 is retained for didactic reasons.) Once the field equations generated by (1.4) and their solutions have been considered it is a straightforward problem to deal with a much wider class of Lagrangian along similar lines; but it turns out that all such Lagrangians are *in effect* no more general than (1.4).

2. Inhomogeneous quadratic Lagrangians

It is helpful to have some of the explicit concomitants of (1.3) available. To this end write $(x, y, u, v) \equiv (x^1, x^2, x^3, x^4)$. Then

$$g^{11} = g^{22} = g^{34} = g^{43} = -1, \qquad g^{33} = 2H, \tag{2.1}$$

and all other components of g^{ij} vanish. The Christoffel symbols are

$$\Gamma^1{}_{44} = -H_x, \quad \Gamma^2{}_{44} = -H_y, \quad \Gamma^3{}_{14} = \Gamma^3{}_{41} = H_x, \quad \Gamma^3{}_{24} = \Gamma^3{}_{42} = H_y, \quad \Gamma^3{}_{44} = H_u, \tag{2.2}$$

subscripts x, y, u, v denoting partial derivatives. The covariant Riemann tensor has only three constituents, i.e. essentially distinct components which do not necessarily vanish:

$$R_{1414} = H_{xx}, \qquad R_{1424} = H_{xy}, \qquad R_{2424} = H_{yy}. \tag{2.3}$$

All other components either vanish or are related to (2.3) through the symmetries of R_{ijkl}. From (2.1), (2.3)

$$R_{44} = H_{xx} + H_{yy} =: \Delta H, \qquad R = 0, \tag{2.4a,b}$$

whilst all other components of R_{ij} vanish. The Weyl tensor has two constituents:

$$C_{1414} = \tfrac{1}{2}(H_{xx} - H_{yy}), \qquad C_{1424} = H_{xy}. \tag{2.5}$$

Now, when forming the functional derivative P^{ij} of the Lagrangian L^* given by (1.4), the term βR^2 will contribute nothing because of (2.4b). Taking into account that $R_{ij}R^{ij} = 0$ also, one finds that

$$P_{ij} = \tfrac{1}{2}\lambda g_{ij} - \eta R_{ij} + \gamma(\Box R_{ij} - 2R_{imnj}R^{mn}). \tag{2.6}$$

The last term on the right vanishes since R^{33} is the only non-zero component of R^{mn} and $R_{i33j} = 0$. To deal with $\Box R_{ij} := g^{mn} R_{ij;mn}$ consider first $R_{ij;m}$. Bearing in mind that only $R_{44} \neq 0$ whilst $\Gamma^4{}_{pq} = 0$, one has $R_{ij;m} = R_{ij,m}$. Then by the same token $R_{ij;mn} = R_{ij,mn} - \Gamma^s{}_{mn} R_{ij,s}$ which always vanishes except when $i = j = 4$. Transvection with g^{mn} now leads to the result

$$\Box R_{44} = -\Delta R_{44}, \tag{2.7}$$

whilst all other $\Box R_{mn}$ vanish. The equations $P_{ij} = 0$ now show that unless $\lambda = 0$ one has an inconsistency. Further, since all P_{ij} except P_{44} vanish identically, the field equations reduce to

$$\eta \Delta H + \gamma \Delta\Delta H = 0, \tag{2.8}$$

granted that λ has been taken to be zero from the outset. This equation is equivalent to

$$\gamma \Delta H + \eta H = \Phi, \tag{2.9}$$

where Φ is an arbitrary harmonic function of x and y which also depends arbitrarily on u.

If $x =: r\cos\theta$, $y =: r\sin\theta$ and $\eta/\gamma =: \kappa^2$, where κ^2 can be negative, the solution of (2.8) may be exhibited in the form

$$H = \mathrm{Re}\left([\mathscr{C}_0(\kappa r) + a_0 \ln r + b_0] + \sum_{n=1}^{\infty} [\mathscr{C}_n(\kappa r) + a_n r^n + b_n r^{-n}]\,\mathrm{e}^{in\theta}\right) \tag{2.10}$$

where the a_s and b_s are arbitrary complex 'constants' and $\mathscr{C}_m(\kappa r)$ is a cylinder function, i.e. a linear sum, with arbitrary complex 'constant' coefficients of two linearly independent solutions of Bessel's equation of order m; 'constancy' of coefficients meaning that they depend on u alone.

As a special case, choose the various 'constants' in (2.10) so that H consists simply of the first term of the sum and suppose that $k^2 := -\kappa^2 > 0$. Then, with $kr =: \rho$,

$$H = \mathrm{Re}\{[aK_1(\rho) + bI_1(\rho) + c\rho + d\rho^{-1}]\,\mathrm{e}^{i\theta}\}.$$

Finally choosing $d = -a/k$, $b = c = 0$, this becomes

$$H = [A(u)\cos\theta + B(u)\sin\theta][K_1(\rho) - \rho^{-1}], \tag{2.11}$$

where $A(u)$ and $B(u)$ are arbitrary real functions of u alone. Since

$$h(\rho) := K_1(\rho) - \rho^{-1}\begin{cases} = \rho \ln\rho + \mathrm{O}(\rho), & \rho \to 0, \\ \sim -\rho^{-1} + \mathrm{O}(\mathrm{e}^{-\rho}/\rho), & \rho \to \infty, \end{cases}$$

H is everywhere finite in this case. (The derivatives with respect to x and y of $h(\rho)$, however, diverge as $\rho \to 0$.) This situation may be contrasted with that which obtains when $\gamma = 0$ or $\eta = 0$: for general values of θ and u, H then necessarily diverges as $\rho \to 0$ or $\rho \to \infty$ or both.

3. Elementary invariants of degree *m*

It will be convenient to adopt the following definition: an invariant $K^{(m)}$ is an elementary invariant of degree m if it is an invariant transvection of m components of the Riemann tensor with $2m$ components of the metric tensor. Thus, generically,

$$K^{(m)} = \prod_{r=1}^{m} R_{m_r n_r p_r q_r} g^{s_r t_r} g^{u_r v_r}, \tag{3.1}$$

where the set of superscripts differs from the set of subscripts only with respect to order. Now write

$$Z^{(m)ijkl} := \partial K^{(m)}/\partial R_{ijkl}, \tag{3.2}$$

where in forming the derivative on the right symmetries of R_{ijkl} are to be ignored.

Thus, if the Riemann tensor be varied, the metric being kept fixed, $\delta K^{(m)}$ will consist of m terms generically all alike and therefore

$$Z^{(m)ijkl} = \sum \prod_{r=1}^{m-1} R_{m_r n_r p_r q_r} g^{s_r t_r} g^{u_r v_r} g^{s_m t_m} g^{u_m v_m} \tag{3.3}$$

where each of the m sets of $4m$ superscripts in the m terms of the sum is some permutation of $m_1 n_1 \ldots p_{m-1} q_{m-1} ijkl$.

Let $T_s^{(m)}$ stand for the sth term of the sum on the right of (3.3). Then, bearing (2.3) in mind, every non-vanishing factor $R_{m_r n_r p_r q_r}$ can only be of the kind $\pm R_{4A_r 4B_r}$, where the range of upper case indices is 1, 2. Since all subscripts are dummy indices, each subscript 4 must be paired with a factor g^{43}, since $g^{4b}=0$ unless $b=3$. $T_s^{(m)}$ thus contains a factor $(g^{43})^{2m-2}$. The superscript 3 occurs $2m-2$ times in this, so that, since the valence of Z^{ijkl} is 4, at most four of these superscripts can be free. Thus there are at least $2m-6$ dummy superscripts which take the (only possible) value 3. All these must be paired with subscripts of the factors $R_{4A_r 4B_r}$ which take the value 3, but there are no such subscripts. It follows that $T_r^{(m)}=0$ when $m>3$, i.e.

$$Z^{ijkl}=0 \qquad \text{when } m>3. \tag{3.4}$$

Since the cases $m=1$ and $m=2$ were in effect disposed of in § 2, it remains to examine the case $m=3$. It is evident from the argument leading up to (3.3) that Z^{ijkl} can have at most one component which does not necessarily vanish, namely Z^{3333}.

When one factor $g^{\cdot\cdot}$ is omitted from the product on the right of (3.1)—it may be taken to be $g^{u_m v_m}$ without loss of generality—one has a tensor of valence 2, $Y_{u_m v_m}$, say. Since there are m factors $R_{\ldots\ldots}$, there are $2m$ subscripts 4, of which at most two can be free. Thus there are at least $2m-2$ dummy subscripts 4 and $Y_{u_m v_m}$ must therefore contain a factor $(g^{43})^{2m-2}$. In this each 3 is the only surviving value of a dummy index; but the corresponding subscript cannot have this value in a non-vanishing component of the Riemann tensor. It follows that

$$Y_{ij}=0 \qquad \text{when } m>1. \tag{3.5}$$

Finally the functional derivative $\delta K^{(m)}/\delta g_{ij}$ is to be calculated. It is convenient temporarily to suppress the index (m), the degree of K so being left understood. Then under a variation of the metric tensor

$$\delta \int K(-g)^{1/2}\, \mathrm{d}^4x = \int (-g)^{1/2} \delta K\, \mathrm{d}^4x$$

since K vanishes. From (3.1), if $Z^{ij} := \partial K/\partial g_{ij}$ (R_{ijkl} being kept fixed in forming this derivative) one has

$$\begin{aligned}\delta K &= Z^{ijkl}\delta R_{ijkl} + Z^{ij}\delta g_{ij} \\ &= -Z_p{}^{jkl}\delta\Gamma^p{}_{jk;l} + (R^i{}_{pkl}Z^{ipkl} + Z^{ij})\delta g_{ij},\end{aligned} \tag{3.6}$$

granted that Z^{ijkl} is taken as skew-symmetric in i, j and in k, l. Thus

$$\delta K = g_{pi} Z^{ijkl}{}_{;l}\delta\Gamma^p{}_{jk} + (R^i{}_{pkl}Z^{ipkl} + Z^{ij})\delta g_{ij} + \text{divergence}. \tag{3.7}$$

Since the variations in (3.6) are supposed to vanish on the boundary, the divergence in (3.7) may be rejected.

Now, Z^{ij} consists of a sum of terms each of which has a factor Y_{uv}, u, v being generic indices. By (3.5), Z^{ij} therefore vanishes when $m>1$. As regards the remaining terms of (3.7), these vanish when $m>3$ by virtue of (3.4). When $m=3$ there can be only one component of $Z^{ijkl}{}_{;l}$ which does not necessarily vanish, namely $Z^{3333}{}_{;3}$. This, however, vanishes because of (2.2) and because K does not depend on x^3. In short

$$\delta K^{(m)}/\delta g_{ij}=0 \qquad \text{when } m>2. \tag{3.8}$$

It may be remarked that (3.8) also follows directly, without explicit reference to Y_{uv}, from a general formula for the functional derivatives of invariants of the Riemann tensor (Buchdahl 1948b).

4. A general class of Lagrangians

It is now a straightforward matter to go on to a much wider class of Lagrangians than that considered in § 2. It consists of Lagrangians which are functions of N elementary invariants: occasionally abbreviating $K_\alpha^{(m_\alpha)}$ to z_α,

$$L=f(z_1,z_2,\ldots,z_N), \tag{4.1}$$

where f can be any function of the z_α which is C^1 in a neighbourhood of the origin and vanishes there:

$$f(0,0,\ldots,0)=0. \tag{4.2}$$

In the case of (1.4) it is this condition which requires the vanishing of λ. Quite generally, if (4.2) is not satisfied the equations $\delta L/\delta g_{ij}=0$ lead to an inconsistency.

The various invariants which enter into (4.1) may be mutually dependent—indeed, when $N>14$ they will certainly be so—but this is of no consequence here.

Now write f_α for the derivative $\partial f/\partial z_\alpha$, taken at the origin. Then, since all the z_α vanish here,

$$\delta\int L(-g)^{1/2}\,\mathrm{d}^4x=\int(-g)^{1/2}\delta L\,\mathrm{d}^4x=\int(-g)^{1/2}\sum_\alpha f_\alpha\delta z_\alpha\,\mathrm{d}^4x,$$

which shows that

$$\delta L/\delta g_{ij}=\sum_\alpha f_\alpha\delta K_\alpha^{(m_\alpha)}/\delta g_{ij}. \tag{4.3}$$

However, because of (3.8), the only terms of the sum which survive are those which have $m_\alpha=1$ or 2; and if these are actually to be present one must require the derivatives f_α in question to be non-zero. In short, in place of the Lagrangian (4.1) one is *effectively* left with the class of inhomogeneous quadratic Lagrangians already dealt with in § 2.

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